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# A stochastic theory of grinding

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**Abstract.** A statistical formulation is developed for the number of particles in a given size range following a grinding action carried out over a period of time. The regeneration point method first used by Janossy in the study of cosmic rays is employed. Essentially, the method is based on the backward form of the Chapman–Kolmogoroff equation and is closely related to the theory of fluctuations in nuclear reactors. A probability balance equation is derived and converted to a more convenient form using a generating function. Some new multi-particle breakup functions are introduced and their properties discussed. It is shown that the mean value equation is identical to that conventionally used for grinding but the equations for the variance and higher moments are new. In a special case, we are able to solve the nonlinear, partial integro-differential equation for the generating function and construct the complete probability distribution of the particle number in a given size range. The method can also be employed to study fibre breakup, of interest in the paper industry, and polymer degradation; it therefore has a wide range of application.

## 1. Introduction

Substantial advances have been made in the theory of grinding and crushing in recent years. The review by Austin (1971) whilst relatively old, remains an excellent introduction to the subject. There are, in fact, two distinct aspects of the grinding problem, which we pose as follows: given an initial collection of particles with a prescribed size distribution, what is the new size distribution after a specified grinding time? In order to solve this problem it is first necessary to know how an individual particle breaks into its component parts due to a single grinding action. That is to say, how many fragments are produced and what is their size distribution. With this knowledge, it is then necessary to calculate the sum total effect due to many such grinding actions in various stages of comminution.

As far as the single grinding action is concerned, we need to construct a function $\omega(v, v')\,dv'$ which is the probability that a particle of initial volume $v$ will, after a grinding action, lead to a number of new particles with volumes distributed in the range $(v', v'+dv')$. Some early work in this area can be found in Epstein (1947) and Kottler (1950). Both of these authors also make some attempts to sum the multi-breakage processes in an effort to prove that the distribution of volumes becomes log–normal in the long time limit. A recent review of fragmentation models is to be found in Englman (1991), with specific details in Derrida and Flyvbjerg (1987).

As far as the evolution of the size distribution with time is concerned, Fillipov (1961) developed a method originally proposed by Kolmogoroff (1941), in which the process of breakage is assumed to be essentially stochastic. Then, using the theory of Markov processes, Fillipov derives equations for the mean value and the variance of the size distribution as a function of grinding time. Subsequent work has concentrated on solving the grinding equation for the mean value using self-similar solutions. The work of Kapur (1972), Gupta and Kapur (1975), Peterson and Scotto (1985) and Peterson (1986) should be noted in this respect. However, it is important to observe that the processes of breakup also occur in other areas of science and engineering, e.g. in floc breakage in agitated suspensions (Pandya and Spielman 1983) and in fibre and polymer breakup (Goren 1968, Ziff and McGrady 1986). The work of Goren, which deals with the distribution lengths in the breakage of fibres or linear polymers, also employs self-similarity. However, Pandya and Spielman actually solve the balance equation for the volume distribution of flocs numerically and compare their results with experiment. Ziff and McGrady offer some explicit analytic solutions for some very simple models of breakup as do Bak and Bak (1959) and Meyer *et al* (1966). A further advance was made by Williams (1990) who was able to solve exactly the grinding equation for the mean value for a general class of grinding functions and breakage rates using methods originally developed in neutron transport theory.

The purpose of the present work is to complement, clarify and extend the work of Filippov (1961). The importance of Filippov's work cannot be over-stated since he established a fundamental approach which enables mean values and the fluctuations about that mean to be studied. He also observed the curious 'loss of mass' phenomenon which has puzzled a number of workers in this field and in some analogous fields (Corngold and Williams, 1991). We reconsider the stochastic formulation of grinding in terms of the backward form of the Chapman–Kolmogoroff equations. Such a formulation has a number of advantages, not least being that it illustrates the physics of the problem very clearly. It also clarifies the role of the breakup function and highlights the approximations inherent in the standard approach based on mean values.

## 2. A simple stochastic model of grinding

This paper is devoted to the development of a general theory of grinding using methods of probability balance based on the regeneration point method. However, because that theory is very general, we feel that it will be useful to outline the approach in a simple way to set the scene for the generalization to come. In order to do this, we consider a particle which on each grinding action can break into two pieces, i.e. binary breakup. To be more precise, if the initial particle has a volume $v$, then after breakup there will be two particles of volumes $v_1$ and $v_2$ such that $v = v_1 + v_2$. The actual magnitudes of $v_1$ and $v_2$ are governed by a probability distribution which, for the sake of example, we take to be uniform. Thus the product particles will have volumes lying with equal probability between zero and the maximum value $v$. After some time, a particle size distribution will build up in the sense that $H(v_0 \to v, t)\,\mathrm{d}v$ will be the number of particles in the size range $v$ to $v + \mathrm{d}v$ at time $t$, if there was one particle of volume $v_0$ at $t = 0$. Associated with this is a fluctuation, arising from the fact that particles are created randomly in size and at random times. Our purpose is to develop a method for calculating the probability distribution associated with that randomness.

The complete details are given in the next section but, to illustrate the procedure, we consider how to calculate the probability distribution associated with the total number of particles present at a given time $t$ after the commencement of grinding. We also assume that the grinding process is characterized by a grinding rate $\phi$ such that $\phi\exp(-\phi(t - t'))\,\mathrm{d}t'$ is the probability that a particle will experience a grinding action in $\mathrm{d}t'$ and undergo no further grinding action in the subsequent time $t - t'$. If we further define $p_n(v, t)$ as the probability that there are $n$ particles present (regardless of size) at time $t$, when there was one particle of volume $v$ at $t = 0$, then

$$p_n(v,t) = \delta_{n,1}\mathrm{e}^{-\phi t} + \phi(v)\int_0^t \mathrm{d}t'\,\mathrm{e}^{-\phi(v)(t-t')} \sum_{n=n_1+n_2} \int_0^v \frac{\mathrm{d}v'}{v} p_{n_1}(v',t')p_{n_2}(v-v',t'). \quad (1)$$

The term $\mathrm{d}v'/v$ arises from our assumption about uniform breakup in size.

If we define the generating function $G(v|Z,t)$ such that

$$G(v|Z,t) = \sum_{n=0}^{\infty} Z^n p_n(v,t) \quad (2)$$

and then multiply equation (1) by $Z^n$ and sum over $n$, the result reduces to

$$G(v|Z,t) = Z\mathrm{e}^{-\phi t} + \int_0^t \mathrm{d}t'\,\mathrm{e}^{-\phi(v)(t-t')} \sum_{n=n_1+n_2} \int_0^v \frac{\mathrm{d}v'}{v} G(v'|Z,t')G(v-v'|Z,t'). \quad (3)$$

It is clear that $p_n(v,0) = \delta_{n,1}$, i.e. at $t = 0$ there is one particle present, and so $G(v|Z,0) = Z$. Differentiation of equation (3) with respect to $t$ simplifies it to

$$\left[\frac{\partial}{\partial t} + \phi(v)\right] G(v|Z,t) = \phi(v)\int_0^v \frac{\mathrm{d}v'}{v} G(v'|Z,t')G(v-v'|Z,t'). \quad (4)$$

This is a nonlinear integro-differential equation which, in general, is difficult to solve. However, in the special case of $\phi(v) = av$, i.e. the grinding rate is proportional to the particle volume, it is readily shown that a solution which satisfies the initial conditions is

$$G(v|Z,t) = Z\mathrm{e}^{avt(Z-1)}. \quad (5)$$

But this is the generating function of the Poisson distribution and leads to

$$p_n(v,t) = \frac{\mathrm{e}^{-avt}(avt)^{n-1}}{(n-1)!} \qquad n \geqslant 1. \quad (6)$$

The mean value of $n$ is $\bar{n} = 1 + avt$ which, as we expect, grows linearly with time at a rate determined by $\phi$. Similarly the variance $\sigma^2 = avt$.

We stress that different forms of $\phi(v)$ lead to other distributions for $p_n$.

## 3. A stochastic formulation of grinding

We base our approach to the grinding problem on a probability balance method devised by Janossy (Bharucha–Reid 1960) for cosmic-ray-shower distributions. Janossy described this approach as the regeneration point or first collision method. It was subsequently extended to cover problems in neutron transport theory by Pal (1961) and by Bell (1965) and also to radiation damage by Williams (1977a).

Let $P_n(v|R,t)$ be the probability that, if there is one particle of volume $v$ at $t = 0$, there will, as a result of grinding, be $n$ particles with volumes lying in the volume range

$R = (v_1, v_2)$ at time $t$ later. We also introduce $\phi(v)$ as the grinding rate [$s^{-1}$] for particles of volume $v$. We further define

$$\chi_i(v \to v_1, v_2, \ldots, v_i)\, dv_1\, dv_2 \ldots dv_i \tag{7}$$

as the probability that the breakage of a particle of volume $v$ leads instantaneously to $i$ new particles with volume in the ranges $(v_1, v_1 + dv_1)$, $(v_2, v_2 + dv_2)$, $\ldots$, $(v_i, v_i + dv_i)$.

Finally, we define the $\Delta$ function as

$$\Delta(v \in R) = \begin{cases} 1 & v \in R \\ 0 & v \notin R. \end{cases} \tag{8}$$

We now use Janossy's first collision method to formulate a balance equation for $P_n$. We write

$$P_n(v|R, t) = \Delta(v \in R)\delta_{n,1} e^{-\phi(v)t} + \phi(v) \int_0^t dt'\, e^{-\phi(v)(t-t')}[II + III + IV + \cdots] \tag{9}$$

we set

$$I = \Delta(v \in R)\delta_{n,1} e^{-\phi(v)t} \tag{10}$$

and define

$$II = \int dv_1\, \chi_1(v \to v_1) P_n(v|R, t') \tag{11}$$

$$III = \int dv_1 \int dv_2\, \chi_2(v \to v_1, v_2) \sum_{n_1+n_2=n} P_{n_1}(v|R, t') P_{n_2}(v|R, t') \tag{12}$$

$$IV = \int dv_1 \int dv_2 \int dv_3\, \chi_3(v \to v_1, v_2, v_3) \sum_{n_1+n_2+n_3=n} P_{n_1}(v|R, t') P_{n_2}(v|R, t') P_{n_3}(v|R, t') \tag{13}$$

etc.

The physical meaning of these terms is as follows. $I$ is the probability that no grinding action takes place, i.e. the initial particle is unchanged. To explain $II$, $III$ and $IV$, etc, we recall that

$$\phi(v) e^{-\phi(v)(t-t')}\, dt' \tag{14}$$

is the probability that a particle will experience a grinding action in $dt'$ and undergo no further grinding action in the subsequent time $t - t'$. Thus terms $II$, $III$, $IV$, etc, are the probabilities that, as a result of grinding, 1, 2, 3, etc new particles will be produced. That is to say, at some general time $t'$, as a result of $n_1$ particles of volume $v_1$, $n_2$ particles of volume $v_2$, etc, such that $v_1 + v_2 + \cdots = v$, and $n_1 + n_2 + \cdots = n$, there will still be $n$ particles in $R$ at a time $t$ later. Each of the events, $I$, $II$, $III$, $IV$, etc are mutually exclusive and so must be added to obtain $P_n$.

The probability balance equation (9) can be written more concisely in terms of the generating function

$$G(v|Z, R, t) = \sum_{n=0}^{\infty} Z^n P_n(v|R, t). \tag{15}$$

Multiplying equation (9) by $Z^n$ and summing over $n$, leads to

$$G(v|Z, R, t) = \Delta(v \in R)Ze^{-\phi(v)t} + \phi(v)\int_0^t dt'\, e^{-\phi(v)(t-t')}$$
$$\times\left[\int dv_1\, \chi_1(v \to v_1)G(v_1|Z, R, t')\right.$$
$$+\int dv_1 \int dv_2\, \chi_2(v \to v_1, v_2)G(v_1|Z, R, t')G(v_2|Z, R, t')$$
$$+\int dv_1 \int dv_2 \int dv_3\, \chi_3(v \to v_1, v_2, v_3)$$
$$\left.\times\, G(v_1|Z, R, t')G(v_2|Z, R, t')G(v_3|Z, R, t') + \cdots\right]. \tag{16}$$

This integral equation can be simplified by differentiating with respect to $t$, leading to

$$\left[\frac{\partial}{\partial t} + \phi(v)\right] G(v|Z, R, t) = \phi(v)\left[\int dv_1\, \chi_1(v \to v_1)G(v_1|Z, R, t)\right.$$
$$+\int dv_1 \int dv_2\, \chi_2(v \to v_1, v_2)G(v_1|Z, R, t)G(v_2|Z, R, t)$$
$$+\int dv_1 \int dv_2 \int dv_3\, \chi_3(v \to v_1, v_2, v_3)$$
$$\left.\times\, G(v_1|Z, R, t)G(v_2|Z, R, t)G(v_3|Z, R, t) + \cdots\right] \tag{17}$$

which is an integro-differential equation for the generating function $G(\ldots)$. There is also an initial condition

$$G(v|Z, R, 0) = \Delta(v \in R)Z. \tag{18}$$

Since an exact solution of equation (17) is unlikely, except in very special circumstances, we shall content ourselves with deriving the equations for the mean value and the variance. From equation (15), we note that

$$\left.\frac{\partial G}{\partial Z}\right|_{Z=1} = \sum_{n=0}^{\infty} nP_n(v|R, t) = \langle N(v|R, t)\rangle \tag{19}$$

$$\left.\frac{\partial^2 G}{\partial Z^2}\right|_{Z=1} = \sum_{n=0}^{\infty} n(n-1)P_n(v|R, t) = \langle N(N-1)\rangle. \tag{20}$$

Thus the variance is

$$\langle N^2\rangle - \langle N\rangle^2 = G''(1) + G'(1) - \{G'(1)\}^2 \tag{21}$$

where we have used an obvious abbreviated notation.

Physically, $\langle N\rangle$ is the mean number of particles at time $t$ in the volume range $R$ given that there was one particle of volume $v$ at $t=0$. More precisely, we may write

$$\langle N(v|R,t)\rangle = \int_R \mathrm{d}v_0\, H(v\to v_0;t) \tag{22}$$

where $H(v\to v_0;t)\,\mathrm{d}v_0$ is the number of particles in the volume range $(v_0, v_0+\mathrm{d}v_0)$ at $t$ due to one particle of volume $v$ at $t=0$. $H(v\to v_0;t)$ plays the role of a Green function as we shall demonstrate below.

The mean square number of particles can be written

$$\langle N(v|R,t)^2\rangle = \langle N(v|R,t)\rangle + \int_R \mathrm{d}v_1 \int_R \mathrm{d}v_2\, n_2(v\to v_1, v_2;t) \tag{23}$$

where $n_2(\ldots)$ is a doublet distribution function.

Differentiating equation (17) with respect to $Z$ and using equation (19), we find

$$\begin{aligned}\frac{\partial\langle N\rangle}{\partial t} + \phi(v)\langle N\rangle = \phi(v)\bigg[\int \chi_1 N_1\,\mathrm{d}v_1 + \iint \chi_2(N_1+N_2)\,\mathrm{d}v_1\,\mathrm{d}v_2 \\ + \iiint \chi_3(N_1+N_2+N_3)\,\mathrm{d}v_1\,\mathrm{d}v_2\,\mathrm{d}v_3 + \cdots\bigg]\end{aligned} \tag{24}$$

where we use the subscript to denote the integration variable. It is clear that, by rearranging the dummy integration variable, we can write equation (24) in the form

$$\left[\frac{\partial}{\partial t}+\phi(v)\right]\langle N(v|R,t)\rangle = p_1\phi(v)\langle N(v|R,t)\rangle + \phi(v)\int_0^v \mathrm{d}v'\,\omega(v,v')\langle N(v'|R,t)\rangle \tag{25}$$

where

$$\begin{aligned}\omega(v,v') = \int \mathrm{d}v_2\,[\chi_2(v\to v',v_2)+\chi_2(v\to v_2,v')] \\ + \int \mathrm{d}v_2 \int \mathrm{d}v_3\,[\chi_3(v\to v',v_2,v_3)+\chi_3(v\to v_2,v',v_3) \\ + \chi_3(v\to v_3,v_2,v')] + \cdots\end{aligned} \tag{26}$$

and we have set $\chi_1(v\to v') = p_1\delta(v-v')$ as required physically. We shall subsequently set $p_1=0$ on the assumption that the particle always breaks on grinding.

Let us now introduce equation (22) into equation (25), whence

$$\left[\frac{\partial}{\partial t}+\phi(v)\right]H(v\to v_0;t) = \phi(v)\int_0^v \mathrm{d}v'\,\omega(v,v')H(v'\to v_0;t). \tag{27}$$

Consider the adjoint of equation (27) with respect to the $v$ variable, namely

$$\left[\frac{\partial}{\partial t}+\phi(v)\right]H^+(v\to v_0;t) = \int_v^{v_0} \mathrm{d}v'\omega(v',v)\phi(v')H^+(v'\to v_0;t). \tag{28}$$

But from the reciprocity theorem (Lanczos 1961), it may be shown that

$$H^+(v\to v_0;t) = H(v_0\to v;t) \tag{29}$$

whence equation (28) becomes

$$\left[\frac{\partial}{\partial t}+\phi(v)\right]H(v_0\to v;t)=\int_v^{v_0}\mathrm{d}v'\omega(v',v)\phi(v')H(v_0\to v';t)+\delta(v-v_0)\delta(t) \tag{30}$$

where we have added the source term to account for the initial condition.

Equation (30) is identical to the grinding equation for the mean number of particles used by earlier workers in the field. The derivation by the method discussed above has the advantage of giving a more detailed definition of the breakup function $\omega(v,'v)$ in terms of its multi-particle components. It also enables us to obtain an expression for the mean square, which we do by differentiating equation (17) twice with respect to $Z$. The result can be written

$$\begin{aligned}\left[\frac{\partial}{\partial t}+\phi(v)\right]V(v|R,t)&=\phi(v)\int_0^v\mathrm{d}v'\,\omega(v,v')V(v'|R,t)+2\phi(v)\\&\times\left[\iint\chi_2N_1N_2\mathrm{d}v_1\,\mathrm{d}v_2+\iiint\chi_3(N_1N_2+N_1N_3+N_2N_3)\,\mathrm{d}v_1\,\mathrm{d}v_2\,\mathrm{d}v_3\right.\\&+\iiiint\chi_4(N_1N_4+N_1N_3+N_1N_2+N_2N_4+N_3N_4+N_2N_3)\,\mathrm{d}v_1\,\mathrm{d}v_2\,\mathrm{d}v_3\,\mathrm{d}v_4\\&\left.+\cdots\right]\end{aligned} \tag{31}$$

where $V=\langle N(N-1)\rangle$. However, because of equation (25), $V$ in equation (31) can also be replaced by $\langle N^2\rangle$ with different initial condition. By some manipulations, equation (31) may be recast into the following form, namely

$$\begin{aligned}\left[\frac{\partial}{\partial t}+\phi(v)\right]V(v|R,t)&=\phi(v)\int_0^v dv'\,\omega(v,v')V(v'|R,t)\\&+2\phi(v)\int\mathrm{d}v_1\int\mathrm{d}v_2\,\langle N(v_1|R,t)\rangle\langle N(v_2|R,t)\rangle K(v;v_1,v_2)\end{aligned} \tag{32}$$

where

$$\begin{aligned}K(v;v_1,v_2)&=\chi_2(v\to v_1,v_2)+\int\mathrm{d}v_3\,[\chi_3(v\to v_1,v_2,v_3)+\chi_3(v\to v_1,v_3,v_2)\\&+\chi_3(v\to v_3,v_2,v_1)]+\int\mathrm{d}v_3\int\mathrm{d}v_4\,[\chi_4(v\to v_1,v_2,v_3,v_4)\\&+\chi_4(v\to v_1,v_4,v_3,v_2)+\chi_4(v\to v_1,v_3,v_2,v_4)+\chi_4(v\to v_4,v_2,v_3,v_1)\\&+\chi_4(v\to v_3,v_4,v_1,v_2)+\chi_4(v\to v_3,v_2,v_1,v_4)]+\cdots.\end{aligned} \tag{33}$$

Now if we write equation (27) as

$$\left[\frac{\partial}{\partial t}+L^+(v)\right]H(v\to v_0;t)=\delta(v-v_0)\delta(t) \tag{34}$$

then equation (32) may be written

$$\left[\frac{\partial}{\partial t}+L^+(v)\right]V(v|R,t)=\phi(v)Q(v,t) \tag{35}$$

where

$$Q(v,t) = 2\int \mathrm{d}v_1 \int \mathrm{d}v_2 \, \langle N(v_1|R,t)\rangle \langle N(v_2|R,t)\rangle K(v;v_1,v_2). \tag{36}$$

It is quite clear then, from the properties of the Green function, that (Williams 1977b, Pazsit 1987)

$$\langle N(v|R,t)^2\rangle = \langle N(v|R,t)\rangle + \int_0^t \mathrm{d}t_0 \int_0^v \mathrm{d}v_0 \, H(v\to v_0; t-t_0)\phi(v_0)Q(v_0,t_0). \tag{37}$$

We therefore have the variance directly in terms of the one-particle Green function. In this section therefore, we have derived the mean value equation from its stochastic counterpart and shown how to calculate the associated uncertainty in the mean via the variance. We shall explore various aspects of these equations below.

## 4. The multi-particle breakup functions

As indicated above, $\chi_i$ defines the volume distribution when the initial particle breaks up into $i$ separate parts. Now it is clear that $\chi_i$ must obey certain constraints. For example,

$$\chi_1(v\to v_1) = p_1(v)\delta(v-v_1) \tag{38}$$

$$\chi_2(v\to v_1,v_2) = p_2(v)\omega_2(v,v_1)\delta(v-v_1-v_2) \tag{39}$$

$$\chi_3(v\to v_1,v_2,v_3) = p_3(v)\omega_3(v,v_1,v_2)\delta(v-v_1-v_2-v_3) \tag{40}$$

$$\chi_4(v\to v_1,v_2,v_3,v_4) = p_4(v)\omega_4(v,v_1,v_2,v_3)\delta(v-v_1-v_2-v_3-v_4) \tag{41}$$

etc.

We also require that

$$\int \mathrm{d}v_1 \int \mathrm{d}v_2 \ldots \int \mathrm{d}v_i \, \chi_i(v\to v_1,v_2\ldots v_i) = p_i(v) \tag{42}$$

$$\int_0^v \mathrm{d}v_1 \int_0^{v-v_1} \mathrm{d}v_2 \int_0^{v-v_1-v_2} \mathrm{d}v_3 \int_0^{v-v_1-v_2-\ldots v_{i-2}} \mathrm{d}v_{i-1}\omega_i(v,v_1,v_2,v_3,\ldots,v_{i-1}) = 1. \tag{43}$$

Equation (43) implies that

$$\int_0^v \mathrm{d}v' \, v'\omega(v,v') = v \tag{44}$$

and hence that equation (30) leads to

$$\int_0^{v_0} \mathrm{d}v \, vH(v_0\to v;t) = v_0 \tag{45}$$

i.e. conservation of volume in the system.

### 4.1. Approximate kernels

It is convenient in many practical cases (Peterson 1986) to write $\omega_n$ in the following homogeneous form:

$$\omega_2(v, v_1) = \frac{1}{v} f_2\left(\frac{v_1}{v}\right) \tag{46}$$

$$\omega_3(v, v_1, v_2) = \frac{1}{v^2} f_3\left(\frac{v_1}{v}\right) g_3\left(\frac{v_2}{v}\right) \tag{47}$$

$$\omega_4(v, v_1, v_2, v_3) = \frac{1}{v^3} f_4\left(\frac{v_1}{v}\right) g_4\left(\frac{v_2}{v}\right) h_4\left(\frac{v_3}{v}\right). \tag{48}$$

Physically, this implies that the volumes of the broken particles are chosen from the distributions $f$, $g$, $h$ etc, subject to $v = v_1 + v_2 + v_3 + \cdots + v_i$.

To be consistent with equation (43), we must have

$$\int_0^1 \mathrm{d}x\, f_2(x) = 1 \tag{49}$$

$$\int_0^1 \mathrm{d}x\, f_3(x) \int_0^{1-x} \mathrm{d}y\, g_3(y) = 1 \tag{50}$$

$$\int_0^1 \mathrm{d}x\, f_4(x) \int_0^{1-x} \mathrm{d}y\, g_4(y) \int_0^{1-x-y} \mathrm{d}z\, h_4(z) = 1 \tag{51}$$

etc.

If equations (46) to (48) are inserted into equation (26) for $\omega(v, v')$, it is readily shown that

$$\begin{aligned}
\omega(v, v') = {} & \frac{p_2}{v}\left[f_2\left(\frac{v'}{v}\right) + f_2\left(\frac{v - v'}{v}\right)\right] \\
& + \frac{p_3}{v}\left[f_3\left(\frac{v'}{v}\right)\int_0^1 \mathrm{d}x\, g_3(x) + g_3\left(\frac{v'}{v}\right)\int_0^1 \mathrm{d}x\, f_3(x) \right. \\
& \left. + \int_0^1 \mathrm{d}x\, f_3\left(1 - x - \frac{v'}{v}\right) g_3(x)\right] + \cdots \qquad (52) \\
\equiv {} & \frac{1}{v} f\left(\frac{v'}{v}\right) \qquad (53)
\end{aligned}$$

i.e. if the individual $f_i$, $g_i$, $h_i$, etc, are homogeneous, then $v\omega(v, v')$ is expressible in homogeneous form also.

A few special cases are worthy of note. For example, if the particle always breaks into the same number of parts, $n = n_0$, then $p_n = \delta_{n,n0}$ and $\omega(v, v')$ would be given by one term in equation (26). Thus, if $n_0 = 2$, we would have

$$\omega(v, v') = \omega_2(v, v') + \omega_2(v, v - v') \tag{54}$$

and so on.

Suppose that the distribution functions of emitted particles were all uniform, i.e. $f$, $g$, $h$, etc, are constants independent of $v'/v$. Then, taking into account the normalization conditions, we find

$$v\omega(v, v') = \sum_{n=2}^{\infty} p_n n(n-1)\left(1 - \frac{v'}{v}\right)^{n-2}. \tag{55}$$

Suppose further that $p_n$ is given by the Poisson law, such that

$$p_n = \frac{1}{1-(\bar{n}+1)e^{-\bar{n}}}\frac{\bar{n}^n e^{-\bar{n}}}{n!}. \tag{56}$$

The reason for the unusual factor in equation (56) is due to the normalization requirement of equation (44), which leads to

$$\sum_{n=2}^{\infty} p_n = 1. \tag{57}$$

With $p_n$ from equation (56), equation (55) may be summed to give

$$v\omega(v, v') = \frac{\bar{n}^2 \exp[-\bar{n}v'/v]}{1-(\bar{n}+1)e^{-\bar{n}}}. \tag{58}$$

Whether this breakup function has any practical value is not known, but it has a number of interesting features and is algebraically simple. It also resembles closely the case of the randomly broken ring discussed by Englman (1991) in his review.

Another approach to the modelling of the $\chi_i$ is to assume that, on average, the masses of the individual components of the particle are conserved. This may be done by writing for $i \geqslant 2$

$$\chi_i(v \to v_1, v_2, \ldots v_i) = \frac{p_i}{v^i}\prod_{j=1}^{i} g_j\left(\frac{v_j}{v}\right) \qquad \text{where} \qquad 0 \leqslant v_j \leqslant v. \tag{59}$$

If $\chi_i$ is in this form, it leads to a particularly simple form for the generating function equation as defined by equation (17), namely

$$\left[\frac{\partial}{\partial t} + \phi(v)\right] G(v|Z, R, t) = \phi(v)\sum_{i=2}^{\infty} p_i \left[\frac{1}{v}\int_0^v dv' g_i\left(\frac{v'}{v}\right) G(v'|Z, R, t)\right]^i. \tag{60}$$

If we differentiate equation (60) with respect to $Z$, we find

$$\left[\frac{\partial}{\partial t} + \phi(v)\right] \langle N(v|R, t)\rangle = \phi(v)\sum_{i=2}^{\infty} i p_i \int_0^v \frac{dv'}{v} g_i\left(\frac{v'}{v}\right) \langle N(v'|R, t)\rangle. \tag{61}$$

The adjoint Green function associated with equation (61) is

$$\left[\frac{\partial}{\partial t} + \phi(v)\right] H(v_0 \to v; t) = \sum_{i=2}^{\infty} i p_i \int_v^{v_0} \frac{dv'}{v'}\phi(v') g_i\left(\frac{v}{v'}\right) H(v_0 \to v'; t). \tag{62}$$

Since

$$\int_0^{v_0} \mathrm{d}v\, vH(v_0 \to v; t) = v_0 \tag{63}$$

we must have

$$i \int_0^1 \mathrm{d}x\, x g_i(x) = 1. \tag{64}$$

But in order to satisfy equation (42), we must also satisfy

$$\int_0^1 \mathrm{d}x\, g_i(x) = 1. \tag{65}$$

Several experimental results (Pandya and Spielman 1983) have suggested that a reasonable form for $g(x)$ is

$$g_i(x) = A_i x^{\alpha_i} \tag{66}$$

It is then easy to show from equations (64) and (65) that

$$g_i(x) = \frac{1}{i-1} x^{(2-i/)(i-1)} \tag{67}$$

for $i \geqslant 2$. Whence $g_2(x) = 1$, $g_3(x) = 1/2x^{1/2}$, $g_4(x) = 1/3x^{1/3}$, $g_5(x) = 1/4x^{3/4}$ and, for large $i$, $g_i(x) \to 1/ix$. The kernel $\omega(v', v)$ is therefore

$$\omega(v', v) = \frac{1}{v'} \sum_{i=2}^{\infty} \frac{ip_i}{i-1} \left(\frac{v}{v'}\right)^{(2-i)/(i-1)} \tag{68}$$

Unfortunately, this sum cannot be carried out explicitly, even with the Poisson form for $p_i$. Nevertheless, it is instructive to note that the Randolph–Ranjan model (Randolph and Ranjan 1977) predicts

$$\omega(v', v) = \frac{n}{v'} \left(\frac{v}{v'}\right)^{n-2}$$

and Reid's model (Reid 1965) predicts

$$\omega(v'v) = \frac{1}{3v'} \left(\frac{v}{v'}\right)^{-5/3}$$

which are similar in form to equation (68).

## 5. Mean value and variance

In section 3, it was shown that explicit expressions can be given for the mean value $\langle N\rangle$ and the variance

$$\sigma^2 = \langle N^2\rangle - \langle N\rangle^2.$$

Here we explore the form of $\langle N\rangle$ and $\sigma^2$ for some simple models. For simplicity we assume binary breakup of the particles. This is not necessary but eases the algebra without losing the essence of the problem. In this case, from equation (26), we find

$$\omega(v, v') = \frac{1}{v}\{\omega_2(v, v') + \omega_2(v, v - v')\}. \tag{69}$$

Using the approximation introduced in equation (46) leads to

$$\omega(v, v') = \frac{1}{v}\left\{f_2\left(\frac{v'}{v}\right) + f_2\left(\frac{v - v'}{v}\right)\right\}. \tag{70}$$

The equation for the Green function $H$ is therefore

$$\left[\frac{\partial}{\partial t} + \phi(v)\right] H(v_0 \to v; t) = \int_v^{v_0} \frac{dv'}{v'}\phi(v')H(v_0 \to v'; t) \times \left\{f_2\left(\frac{v'}{v}\right) + f_2\left(\frac{v - v'}{v}\right)\right\} + \delta(v - v_0)\delta(t). \tag{71}$$

Assume now that the particle breakup occurs randomly between $(0, v)$ whence $f_2(x) = 1$. Then

$$\left[\frac{\partial}{\partial t} + \phi(v)\right] H(v; t) = 2\int_v^{v_0} \frac{dv'}{v'}\phi(v')H(v'; t) + \delta(v - v_0)\delta(t) \tag{72}$$

where we have suppressed $v_0$ in $H(\ldots)$. Defining the Laplace transform of $H(v, t)$ as

$$\bar{H}(v, s) = \int_0^\infty dt\, e^{-st} H(v, t) \tag{73}$$

we can solve equation (72) to obtain

$$\bar{H}(v, s) = \frac{\delta(v - v_0)}{s + \phi(v_0)} + \frac{2\phi(v_0)}{v_0(s + \phi(v_0))(s + \phi(v))} \exp\left\{2\int_v^{v_0} \frac{\phi(v')\, dv'}{v'(s + \phi(v'))}\right\}. \tag{74}$$

In the special case of $\phi(v) = av$, we can evaluate the integral in equation (74) and perform the Laplace inversion to get

$$H(v_0 \to v; t) = \delta(v - v_0)e^{-av_0t} + at[2 + a(v_0 - v)t]e^{-avt}. \tag{75}$$

It is readily verified that

$$\int_0^{v_0} dv\, vH(v_0 \to v; t) = v_0 \tag{76}$$

as expected from conservation. Similarly, the average number of particles at time $t$ is

$$N(t) = \int_0^v \mathrm{d}v_0\, H(v \to v_0; t) = 1 + avt \tag{77}$$

i.e. a linear increase as grinding proceeds. $N(t)$ is actually $\langle N(v|R,t)\rangle$ when $R = (0, v)$. If we wish to calculate $\langle N^2 \rangle$, we return to equation (37) and note that

$$Q(v,t) = 2\int \mathrm{d}v_1 \int \mathrm{d}v_2 \langle N(v_1|R,t)\rangle\langle N(v_2|R,t)\rangle \chi_2(v \to v_1, v_2) \tag{78}$$

where

$$\begin{aligned}\chi_2(v \to v_1, v_2) &= \omega_2(v, v_1)\delta(v - v_1 - v_2)\\ &= \frac{1}{v} f_2\left(\frac{v_1}{v}\right)\delta(v - v_1 - v_2).\end{aligned} \tag{79}$$

But if $R = (0, v)$, we find

$$\begin{aligned}Q(v,t) &= 2\int \mathrm{d}v_1 \int \mathrm{d}v_2\, (1 + av_1t)(1 + av_2t)\frac{1}{v}\delta(v - v_1 - v_2)\\ &= 2\int_0^v \frac{\mathrm{d}v_1}{v}(1 + av_1t)(1 + (v - v_1)at)\\ &= 2[1 + avt + \tfrac{1}{6}a^2v^2t^2].\end{aligned} \tag{80}$$

From which using equation (37)

$$\langle N(t)^2\rangle = \langle N(t)\rangle + 2\int_0^t \mathrm{d}t_0 \int_0^v \mathrm{d}v_0\, H(v \to v_0; t - t_0)av_0[1 + av_0t_0 + \tfrac{1}{6}a^2v_0^2t_0^2]. \tag{81}$$

After some tedious but simple algebra, we obtain

$$\sigma^2(t) = \langle N(t)^2\rangle - \langle N(t)\rangle^2 = avt. \tag{82}$$

(This integration was checked by MATHEMATICA.)

The fact that the variance and mean are related by

$$\langle N\rangle = 1 + \sigma^2 \tag{83}$$

suggests that the process is Poisson. In fact, if we return to equation (17) and write it for binary splitting, we obtain

$$\left[\frac{\partial}{\partial t} + \phi(v)\right] G(v,t) = \phi(v)\int_0^v \frac{\mathrm{d}v'}{v} f\left(\frac{v'}{v}\right) G(v',t)G(v - v',t) \tag{84}$$

where we have abbreviated $G(v|Z,R,t)$ as $G(v,t)$. If we set $f = 1$ and $\phi = av$, equation (78) reduces to

$$\left[\frac{\partial}{\partial t} + av\right] G(v,t) = a\int_0^v \mathrm{d}v'\, G(v',t)G(v - v',t). \tag{85}$$

This is a nonlinear, integro-differential equation, but one which can be solved exactly as follows. Define the Laplace transform of $G$ as

$$\bar{G}(p,t) = \int_0^\infty \mathrm{d}v\, \mathrm{e}^{-vp} G(v,t)$$

and apply it to equation (85). It is readily seen that the equation reduces to

$$\left[\frac{\partial}{\partial t} - a\frac{\partial}{\partial p}\right]\bar{G}(p,t) = a\bar{G}^2(p,t). \tag{86}$$

This partial differential equation can be solved by the method of characteristics (Sneddon 1957) and leads to

$$\bar{G}(p,t) = \frac{Z[\mathrm{e}^{-(p+at)v_1} - \mathrm{e}^{-(p+at)v_2}]}{p + at - Zat[\mathrm{e}^{-(p+at)v_1} - \mathrm{e}^{-(p+at)v_2}]}. \tag{87}$$

For the initial condition, which is

$$G(v,0) = \Delta(v \in R)Z \tag{88}$$

we have set $R = (v_1, v_2)$, i.e. some volume range in the overall range $(0, v)$. Equation (87) is amenable to Laplace inversion but for simplicity, we take $v_1 = 0$ and $v_2 = \infty$, when

$$G(v,t) = Z\mathrm{e}^{avt(Z-1)} \tag{89}$$

which is indeed the generating function of a Poisson distribution and agrees with the simple results in section 2.

## 6. Summary and conclusions

The grinding of material is essentially a random process. For this reason we have gone beyond the usual investigations of grinding, which deal only with the mean value, to show how a quite general equation for the probability distribution of the number of particles in a given size range may be calculated. A probability balance equation is constructed and converted to an equation for a generating function from which the mean, variance and higher statistical moments can be calculated. In carrying out the formulation, it has been necessary to define new functions describing the probability of breakup of a particle into a number of smaller parts. These multi-particle breakup functions have to satisfy certain conservation conditions and we have elaborated on these.

A particularly useful measure of the fluctuations can be obtained from the variance. We have constructed an equation for this and shown how its solution can be obtained in terms of a one-particle Green function which also defines the mean value. In fact, all higher moments can ultimately be defined in terms of this Green function. Such a procedure was first noted in the neutron transport field and later for radiation damage. The present paper demonstrates once again the power of the method in grinding dynamics.

We have also discussed some possible new breakup functions using multi-particle breakage based on a Poisson distribution. A rather simple model of breakup which we call the statistical model, also seems useful and relates in general terms to models already proposed on phenomenological grounds by Randolph and Ranjan and by Reid.

As an example of the statistical fluctuations, we consider a model of breakup in which the particle breaks into two parts at each grinding action. Furthermore, we assume that the probability of breaking into a given size range is uniform, i.e. any size in the range up to the initial size is equally likely. Finally, we assume that the grinding rate is proportional to particle size. This model, which is not unrealistic, allows a complete solution for the generating function and hence for the probability distribution. In the special case of the total number of particles present in the system, regardless of size, we have derived this probability distribution and shown it to obey the Poisson law. Other forms of breakup function and grinding rate will lead to non-Poisson statistics which can be obtained using the methods developed here.

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